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Allocation of Risk Capital Based on Iso-Entropic Coherent Risk Measure

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Abstract:

Purpose: The potential of diversified portfolio leads to the risk capital allocation problem. There are many kinds of methods or rules to allocate risk capital. However, they have flaws, such as non-continuity, unfairness. In order to get a better method, we propose a new risk measure to be the base of risk capital allocation rule.

Design/methodology/approach: We proposed two kinds of allocation methods: one is marginal risk contribution based on iso-entropic coherent risk measure(IE), the other one is to combine the minimal excess allocation(EBA) principle and IE into risk capital allocation. The iso-entropic coherent risk measure has many advantages over others; it is continuous and more powerful in distinguishing risks, consistent with higher-order stochastic dominances than other risk measures. And EBA is consistent with the amount of risk, which means fairness for risk capital allocation.

Findings: Through cases, simulations and empirical application, it shows that these two allocation rules satisfy some good properties, can be more efficient, more precise and fairer. And the EBA based on IE may be the better one.

Research limitations/implications: However, there are some problems still open. One is how to treat the negative value of allocation. Second is that the consistence between the allocated risk capital and the amount of the risk needs to be studied further.

Originality/value: A good risk measure is very important for risk capital allocation. We proposed two methods to deal with risk capital allocation based on a new coherent risk measure called iso-entropic risk measure, which is smooth and consistent with higher-order stochastic dominance and has higher resolution of risk. It shows that the risk capital allocation rules based on iso-entropic risk measure are better than the other rules.

Keywords: risk capital allocation, marginal risk contribution, minimal excess principle, iso-entropic coherent risk measure

1. Introduction

A financial institution faced with possibility of loss must hold an amount of money(risk capital) to act as a buffer for the risky position. The risky position portfolio is composed of different subportfolios, divisions, or lines of business. Because they are not perfectly correlated, some hedge potential may come from the composed portfolio, which means that the total risk capital for the composed portfolio can be less than the sum of all the risk capital of subportfolios. How to allocate the total risk capital to each subportfolios fairly is very important, because it is the key for performance evaluation as well as for pricing decisions (Van Gerwald, De Waegenaere & Norde, 2012).

There are many papers involving this problem of risk capital allocation. Tasche (1999) argues that the only ''appropriate'' way to allocate risk capital is to determine the marginal risk contribution of each subportfolio. Cummins (2000) reviews different allocation methods in insurance industry, he argues that the methods must be associated with instruments for management decision. Overbeck (2000) apply the marginal risk contribution method based on expected shortfall (ES) computation to allocate risk capital in a credit portfolio. Myers and Read (2001) discuss the marginal risk contribution allocation rule based on game theory. Denault (2001) argues that the essential of allocation is to allocate the benefit from the diversification of portfolio into every sub-portfolio fairly. In the framework of game theory, he shows that when business divisions are infinitely divisible, the only allocation that satisfies this fairness condition is the marginal risk contribution, called Aumann–Shapley value. Overbeck (2000) and Panjer (2002) propose a kind of marginal risk contribution allocation rule based on risk measure of Expected Shortfall (ES). This is called Conditional Tail Expectation (CTE) rule, which is a special case of Aumann–Shapley value.

Tasche (2004) studies the marginal risk contribution from the view of risk performance management, and he points out that marginal risk contribution is the derivative of total risk capital. Venter (2004) argues that besides 'fairness', the profitability of every division and its potential benefit to the whole portfolio(company) must be considered in the goal of capital allocation.

Kalkbrener (2005) proposes a system of axioms that a good risk capital allocation rule must satisfy. He points that the suitable allocation method satisfying all the axioms must be based on the risk measure with the property of subadditivity and positive homogeneity. Dhaene, Henrard, Landsman, Vandendorpe and Vanduffel (2008) proposes a kind of allocation rule based on CTE rule, where the distribution of risk is elliptical, and presents the approximate solutions for normal distribution and log-normal distribution. Kim and Hardy (2009) suggest an allocation rule based on solvent capacity, which satisfies the requirement of the limited liability of shareholders.

Dhaene, Tsankas, Emiliano and Vanduffel (2012) view this allocation problem from another perspective. In their rule, it requires that the weighted sum of difference between real loss and the capital allocated to individual division must be the minimal. Based on this idea, Van Gerwald et al. (2012) propose one kind of risk capital allocation method called Excess-Based Allocation (EBA) rule. Expected excess loss of one subportfolio is the expected of the difference greater than zero between the loss of the subportfolio and the risk capital allocated to it. The goal of their method is to find a solution to minimize the excesses of all the subportfolios in a lexicographical sense (Fishburn, 1974). This approach is inspired by the fact that allocated risk capital based on the Aumann–Shapley value can lead to undesirable result in the sense that the expected excess loss can differ substantially among subportfolios. Large differences in expected excess losses could be perceived as unfair by managers who are evaluated.

Zheng and Chen (2012) propose a new coherent risk measure based on relative entropy, which is obtained under the theory framework of coherent risk measure from Artzner, Delbaen, Eber and Heath (1999). They call this new measure iso-entropic risk measure. And they proved that the new risk measure is smooth and is consistence with higher-order stochastic dominance than ES, so it has higher capacity of identification to risk. The application of it to portfolios selection, it turns out that it has more powerful in identification the risk (Zheng & Chen, 2014a,b).

Based on the iso-entropic risk measure, we propose two kinds of risk capital allocation methods: one is marginal risk contribution (or Aumann-Shapley value) based on iso-entropic coherent risk measure (IE), the other one is to combine the minimal excess allocation (EBA) principle and IE into risk capital allocation. We show that these two allocation rules satisfy some good properties, can be more efficient, more precise and fairer. And the EBA based on IE may be better. The methods are showed through examples and real data case.

The rest of the paper are arranged as follows: in section 2, the iso-entropic coherent risk measure is introduced; in section 3, we introduce two kinds of risk capital allocation rules based on iso-entropic coherent risk measure; and give some examples to show their good properties; in section 4, these risk capital allocation rules are applied to real data in Chinese market; in section 5, the performance of these allocations are displayed; in the last section, we conclude.

2. Iso-entropic Coherent Risk Measure

Here, we introduce briefly the main risk measures used for allocation of risk capital, especially the iso-entropic coherent risk measure proposed by Zheng and Chen (2012).

This paper focuses mainly on the allocation of risk capital to subportfolios. We consider a portfolio consisting of N subportfolios, whose losses are random X_i, i = 1,..., N at a given future data. So the whole portfolio is $X_P = \sum^{N_{i=1}} X_i$. Throughout the paper we use some notations as following:

$\mathbb{N} = \{1, \ldots, N\}$ denotes the set of subportfolios.

$\Omega = \{\omega_{1,} \ldots, \omega_m\}$ denotes the set of states of the world.

$\pi(\omega)$ is the probability of $X_i(\omega)$ in the state ω.

2.1. The Coherent Risk Measures

How to measure the risk of a position is one of the basic tasks in finance. Before allocating the risk capital, the total risk capital requires must be computed by risk measure. The most widely used methods in practice are standard deviation and subsequent VaR(Value at Risk). However, both have serious drawbacks. At the final of last century, a new promising method to quantify risk was proposed by Artzner et al. (1999). Since their seminal work, the theory of coherent risk measures has rapidly been evolving (Acerbi, Nordio & Sirtori, 2001; Acerbi & Tasche, 2002). Recently, the most fashionable coherent risk measure is ES. Compared to VaR, it measures not only the probability of loss but its severity as well. It is seemly that ES might be the most important subclass of coherent risk measures. The four conditions a coherent risk measure must satisfy is as follows:

- Monotonicity: if $X \geq Y$, then $\rho(X) \geq \rho(Y)$.
- Translation invariance: if $c \in \mathbb{R}$, then $\rho(X + c) = \rho(X) + c$.

- Positive Homogeneity: if $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X) \cdot \rho(X + Y) \leq \rho(X) + \rho(Y)$.
- Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Apparently, standard deviation σ as a risk measure is not a coherent one. It does not satisfy monotonicity and translation invariance.

VaR at level $\alpha \in (0,1]$ defined for X on a probability space (Ω, F, P) is:

$$VaR_{\alpha}(X) = q_{1-\alpha}(X) \tag{2.1}$$

where $q_{\alpha}(X)$ is the 1- α -quantile of X. VaR does not satisfy subadditivity, so the total risk capital for the composed portfolio can be more than the sum of all the risk capital of subportfolios. This is not identical to the real intuitive feeling about risk diversification of portfolio, so VaR is not a suitable risk measure for capital allocation.

ES at level $\alpha \in (0,1]$ is defined as:

$$ES_{\alpha}(X) = \frac{1}{\alpha} \int_{0}^{\alpha} VaR_{u}(X) du \tag{2.2}$$

ES satisfies monotonicity, translation invariance and positive homogeneity, and subadditivity, it is coherent.

2.2. Iso-Entropic Coherent Risk Measure

It is seemly that ES might be the most fashionable king of coherent risk measures. However, it has disadvantage that it depends only on the tail of the distribution, so it is not smooth (Cherny & Madan, 2006). And it is consistence only with second-order stochastic dominance, which will lower its capacity of identification to risk.

Zheng and Chen (2012) propose a new coherent risk measure based on relative entropy, which is obtained under the theory framework of coherent risk measure from Artzner et al. (1999). They call this new measure iso-entropic risk measure. And they proved that the new risk measure is smooth and is consistence with higher-order stochastic dominance than ES, so it has higher capacity of identification to risk. The application of it to portfolios selection, it turns out that it has more powerful in identification the risk (Zheng and Chen, 2014a,b).

According to the theorem of Artzner et al. (1999), any coherent risk measure ρ has basic representation as follows:

$$\begin{gathered} \rho(X) = \sup_{Z \in D} e_{P}[ZX] \\ \text{subjected to}: Z \geq 0, E_{P}[Z] = 1 \end{gathered} \tag{2.3}$$

with a certain set D of probability measures absolutely continuous with respect to P. Zheng and Chen (2012) define the set D via relative entropy:

$$D = \{Z : H_\alpha = E_P[Z \log Z]\} \tag{2.4}$$

Denote $Z = {}^{dQ}\!/_{dP}$, then $E_P[Z \log Z]$ is relative entropy of $Q = P$. Relative entropy is also called Kullback–Leibler divergence or information divergence. Denote $H_\alpha = g(\alpha)$, $\alpha \in (0,1]$. By using calculus of variations, the solution is $Z^* = z(X, a) = \frac{e^{mX}}{c}$

where $c = E\,[e^{mX}]$, $E[\cdot] = E_P[\cdot]$ (for simple, subscript is omitted thereafter). At the mean time it must hold that $m \geq 0$ and:

$$E\left[\frac{(mX - \log c)e^{mX}}{c}\right] = H_a \tag{2.5}$$

It means that $m = m(X,a)$ is determined by X and a through formula(2.5) uniquely. Then, the iso-entropic risk measure has the style as follows:

$$\rho_a(X) = E_p[Z^* X] = \int_0^1 q_u(X) z_u(X, a) du \tag{2.6}$$

where $Z_u(X, a) = \frac{e^{mq_u(X)}}{c}$.

Apparently, iso-entropic risk measure is coherent. It satisfies all the four axioms: monotonicity, translation invariance, positive homogeneity and convexity or subadditivity. And because it depends not only on the tail of the distribution, but the whole information, it is not a 0-1 risk measure, so it is smooth.

According to Zheng and Chen (2014a), one good risk measure must have high resolution of risk, besides that it satisfies the four axioms above mentioned. The resolution of risk is the discrimination capacity of risk. The theory of stochastic dominance may be the most important method to test the discrimination capacity of risk measure by now. The relationship between the risk measure and stochastic dominance determines the resolution of risk for the risk measure. In general, the order of risk measure consistent with stochastic dominance determines the resolution of risk, higher the order, higher the resolution is.

It is pointed out that *VaR* is consistent with first-order stochastic dominance but *ES* is consistent with second-order stochastic dominance (Wong & Ma, 2008; Ma & Wong, 2010). Zheng and Chen (2014a) proved that iso-entropic risk measure is consistent with higher-order stochastic dominance as the following theorem:

Theorem 2.1

For all X and Y that have continum supports and are atomless and absolute integrable, and have n-2-order continuous differentiable distribution function then for $\forall a \in (0,1]$, it must hold true that

$$X \overset{nSD}{\precsim} Y \Rightarrow \rho_\alpha(X) \leq \rho_\alpha(Y), \quad \forall \alpha \in (0,1], n=1,2,3... \tag{2.7}$$

Here $X \overset{nSD}{\precsim} Y$ is denotes that X stochastic dominates Y with n-order (simplified as *nSD*).

It must be pointed that when $n > 2$, the condition of continuous differentiable distribution function is not be needed, which means that iso-entropic risk measure are at least consistent with higher-order stochastic dominances the first-order and second-order.

Consistence with stochastic dominance in n-order means that risk measure can rank the two risks X and Y with the same order stochastic dominance, correctly. However, if the two risks X and Y have not the same order stochastic dominance with that of the risk measure, it cannot rank them rightly. So, the iso-entropic risk measure has higher resolution of risk. It is more suitable as a risk measure. Zheng and Chen (2014b) use portfolios selection model based on this iso-entropic risk measure in Chinese stock market, it turns out that this risk measure has the most powerful higher resolution of risk, comparing with other risk measures like standard error, *VaR* and *ES*. In this paper, we will allocate the risk capital based on this iso-entropic risk measure.

3. Risk Capital Allocation Models

In this section, we define risk capital allocation problem and its rules, and then introduce some methods for risk capital allocation, especially the method based on iso-entropic risk measure.

3.1. The Problem of Risk Capital Allocation

Given a coherent risk measure ρ, the risk capital needed is $\rho(X_p) = \rho\left(\sum_{i=1}^{N} X_i\right)$, and according to Subadditivity, it implies that:

$$\rho\left(\sum_{i=1}^{N} X_i\right) \leq \sum_{i=1}^{N} \rho(X_i)$$

This means that the amount of risk capital for the aggregate portfolio is weakly less than the sum of the amounts of risk capital for every individual subportfolio because of diversification effects. There are gains from diversification, and the allocation problem is how to allocate these gains over the subportfolios properly.

Denote that the total risk capital for the portfolio is a_p, for arbitrary allocation $A(R) = \{a_1,\ldots,a_N\}$, it satisfies

$$a_p = \sum_{i=1}^{N} a_i = \rho(X_p) \tag{3.1}$$

Here $R = R(X, \pi, \rho)$ is one certain allocation problem. $\pi(\omega)$ is the probability of $X_i(\omega)$ in the state ω. $\Omega = \{\omega_1,\ldots\omega_m\}$ denotes the set of states of the world. $X = \{X_1,\ldots X_N\}$ denotes the losses of the n subportfolios. ρ is one certain risk measure.

Van Gerwald et al. (2012) argue that a good allocation rule the following properties:

- No Diversification: if $\rho(X_p) = \sum_{i}^{N} \rho(X_i)$, then $a_i = \rho(X_i)$.
- Riskless Portfolio: if $X_i = c$, then $a_i = c$.
- Symmetry: if $X_i = X_j$, then $a_i = a_j$.
- Translation Invariance: if $\hat{X} = \left[X_1, \ldots, X_i + c, X_N\right]$, then $\hat{a}_j = a_j + ce^i$, where e^i is the unit vector for portfolio i.
- Scale Invariance: if $\hat{X} = cX$, then $\hat{a}_j = ca_j$

In the following subsection, we will introduce some methods of risk capital allocation, and their drawbacks, then introduce our methods based on iso-entropic risk measure.

3.2. Marginal Risk Contribution Methods and Iso-Entropic Coherent Risk Measure

The marginal risk contribution rule has received considerable attention in the literature. In game-theoretic framework, it is also called Aumann-Shapley value (Denault, 2001). Let $r : [0,1]^N \rightarrow R$ be given by:

$$r(s_1,\ldots,s_N) = r(s) = \rho\left(\sum_{i=1}^{N} s_i X_i\right) \tag{3.2}$$

The marginal risk contribution (Aumann–Shapley value) is defined as:

$$a_i = \frac{\partial r(s)}{\partial s_i}\Big|_{s=(1,\ldots,1)} \text{ for all } \quad i \in \mathbb{N} \tag{3.3}$$

This means that the risk capital a_i allocated to a subportforlio $i \in \mathbb{N}$ is its marginal risk contribution to the aggregate risk $\rho\left(\sum_{i=1}^{N} s_i X_i\right)$.

A special case is that when the risk measure is Expected Shortfall (Dhaene et al., 2008). In that case, the marginal risk contribution, if it exists, is given by

$$aes_i = \frac{1}{\alpha} E[X_i 1_{Xp > q_{1-\alpha}(X_p)} + \beta X_i 1_{X\,p = q_{1-\alpha}(X_p)}] \tag{3.4}$$

Where $\beta = \begin{cases} \dfrac{\alpha - E[1_{Xp > q_{1-\alpha}(x_p)}]}{E[1_{Xp = q_{1-\alpha}(x_p)}]} & E[1_{Xp = q_{1-\alpha}(x_p)}] > 0. \\ 0, & \text{else} \end{cases}$.

This method is also called to CTE (conditional tail expectation) allocation. When the portfolio X_p is continuous, β = 0, then (3.3) changes into:

$$aes_i = E[X_i \mid X_p > q_{1-\alpha}(X_p)] \tag{3.5}$$

It can be seen that marginal risk contribution rule satisfies all the five properties above mentioned. Denault (2001) and Tasche (1999) identify the marginal risk contribution rule as the unique rule that satisfies some desirable properties.

However, It has been argued that one of the drawbacks of the Aumann-Shapley value is that it requires differentiability of *r(·)* (Fischer, 2003). Van Gerwald et al. (2012) shows that even when the differentiability condition is satisfied, allocating risk capital based on the Aumann-Shapley value may lead to undesirable allocations. For example, when the risk measure is Expected Shortfall, because it only considers a little part of information of the loss distribution, the amount of risk capital allocated to a portfolio depends only on the distribution of its loss in those particular states of the world. This can lead to allocations that the more risky portfolio gets allocated less risk capital and would be perceived as unfair.

For this reason, we propose risk capital allocation methods based on iso-entropic coherent risk measure. Iso-entropic coherent risk measure used all the information of the loss distribution, so the amount of risk capital allocated to a portfolio depends on the distribution of all the states of the world. And, it is not a 0-1 risk measure, it is a smooth one.

According to the marginal risk contribution rule, we can induce the marginal risk contribution rule or the Aumann-Shapley value based on iso-entropic coherent risk measure. From Equation (2.5-2.6) and Equation (3.2), we have

$$r(s_{1,\ldots}, s_N) = r(s) = \frac{E[e^{m_s X_{Ps}} x_{Ps}]}{E e^{m_s X_{Ps}}} \tag{3.6}$$

Where m_s satisfies:

$$E\left[\frac{(m_s X_{Ps} - \log(Ee^{m_s X_{ps}}))e^{m_s X_{Ps}}}{Ee^{m_s X_{Ps}}}\right] = H_\alpha \tag{3.7}$$

and $X_{Ps} = \left(\sum_{i=1}^{N} s_i X_i\right)$.

Through Equation (3.7), Equation (3.6) can be rewritten by:

$$r(s_{1,\ldots}, s_N) = r(s) = \frac{H_\alpha + \log E[e^{m_s X_{Ps}}]}{m_s} \tag{3.8}$$

So we have

$$\frac{\partial r(s)}{\partial s_i} = \frac{E[e^{m_{Ps} X_{Ps}} s_i X_i]}{Ee^{m_{Ps} X_{Ps}}} \tag{3.9}$$

So the marginal contribution based on iso-entropic risk measure is as follows:

$$aie_i = \frac{\partial r(s)}{\partial s_i} \Big|_{s=(1,\ldots,1)} = \frac{E[e^{m_P X_P} X_i]}{Ee^{m_P X_P}} \tag{3.10}$$

Where $X_P = \left(\sum_{i=1}^{N} X_i\right)$, and m_P satisfies:

$$E\left[\frac{(m_P X_P - \log(Ee^{m_P X_P}))e^{m_P X_P}}{Ee^{m_P X_P}}\right] = H_\alpha \tag{3.11}$$

It can be seen that marginal risk contribution rule aie_i satisfies all the five properties above mentioned. This is very easy to be proved.

As above mentioned, because the iso-entropic coherent risk measure is better than *ES*, the Aumann-Shapley value based on iso-entropic coherent risk measure is better than that based on *ES* measure. To show this, we give an example, which is the same as Van Gerwald et al. (2012).

Example 3.1.

Consider a risk capital allocation problem defined by $\Omega = \{\omega_1, \ldots \omega_4\}$, and $\mathbb{N} = \{1,2\}$

$$\pi = \begin{bmatrix} 0,1 \\ 0,1 \\ 0,4 \\ 0,4 \end{bmatrix}, X_1 = \begin{bmatrix} 60 \\ 0 \\ 30 \\ -15 \end{bmatrix}, X_2 = \begin{bmatrix} 6 \\ 60 \\ \gamma \\ 30 \end{bmatrix}$$

where $\gamma \in \mathbb{R}$.

So

$$X_p = X_1 + X_2 = \begin{bmatrix} 66 \\ 60 \\ 30+\gamma \\ 15 \end{bmatrix}$$

Firstly, we compute the expected shortfall (*ES*) and iso-entropic risk measure (*IE*, hereafter) for every subportfolio X_i and portfolio X_P at α = 0.15 level. The distribution of X_1 is known, so its risk amount is fixed with any risk measures, and $ES(X_1)$ = 50, $IE(X_1)$ = 41.6. However, the risk amount of X_2 and X_P depend on parameter $\gamma \in \mathbb{R}$. Their expected shortfall amounts are:

$$ES(X_2) = 50 * 1_{\gamma \le 30} + (40 + \frac{\gamma}{3}) 1_{>30}$$

$$ES(X_p) = 64 * 1_{\gamma \le 30} + (54 + \frac{\gamma}{3}) 1_{\in [30,36]} + (30 + \gamma) 1_{>36}$$

This means that when γ < 30, the risk amount for X_1 and X_P with *ES* measure are fixed. But, it is obvious that the risk amounts will be decreased with the parameter γ decreasing. Because the *ES* risk measure only considers part of the information of the distribution, it can not show this decreases at all! But, the iso-entropic risk measure can show this. We calculate the risk amount $IE(X_2)$ and $IE(X_P)$ with iso-entropic risk measure. Because there are no analytic solution for iso-entropic risk measure, we just compute the numerical value, and the results are showed in Figure 1. From Figure 1, we can see that, $IE(X_2)$ and $IE(X_P)$ vary with parameter γ, which is consistent with the real sense. For example, if γ = -15, X_2 is riskier than X_1, but $ES(X_1)$ = $ES(X_2)$, which means expected shortfall can not distinguish them. However, iso-entropic risk measure can do, because $IE(X_1)$ < $IE(X_2)$.

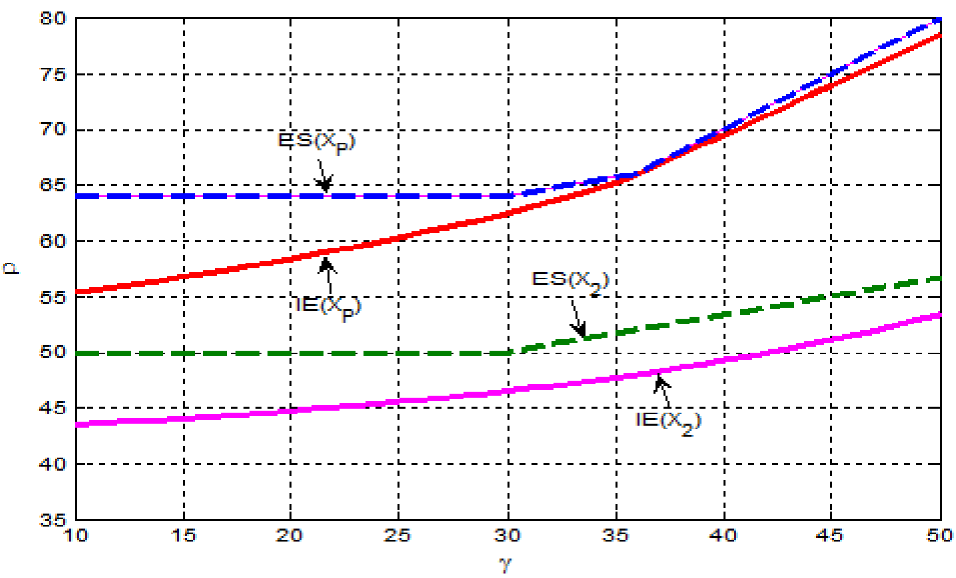

Figure 1. Risk amount dynamics with γ based on different risk measures

Secondly, we compute the marginal risk contribution (Aumann-Shapley value) based on these two risk measures. For expected shortfall (*ES*), according to Equation (3.4), when γ = 30, or γ = 36, $ES(s_1X_1 + s_2X_2)$ is not differentiable, which means that Aumann-Shapley value based on *ES* does not exist at γ = {30,36}. The other Aumann-Shapley values are as follows:

$$aes(X_1)=40*1_{\gamma<30}+50*1_{\gamma<(30,36)}+30*1_{\gamma>36}$$

$$aes(X_2)=24*1_{\gamma<30}+(4+\frac{\gamma}{3})1_{\gamma<(30,36)}+\gamma\, 1_{\gamma>36}$$

As Van Gerwald et al. (2012) pointed, at γ = {30,36}, Aumann-Shapley values for both X_1 and X_2 have significant change. For example, with a marginal increase in γ from just below 30 to just above 30, the amount of risk capital allocated to subportfolio X_2 jumps down by 41.7%; and with a marginal increase in γ from just below 36 to just above 36, the amount of risk capital allocated to subportfolio X_2 jumps up by 125%, it is so big!

But, when we use iso-entropic risk measure to compute Aumann-Shapley values, the indifferentiable point does not exist, there are no such abrupt changes, the Aumann-Shapley values varies with γ smoothly. The results are showed in Figure 2. From Figure 2, we can conclude that iso-entropic risk measure is more suitable for risk capital allocation than expected shortfall. Iso-entropic risk measure can reflect the real change of risk sensitively; and it is smooth, there are no differentiability problems.

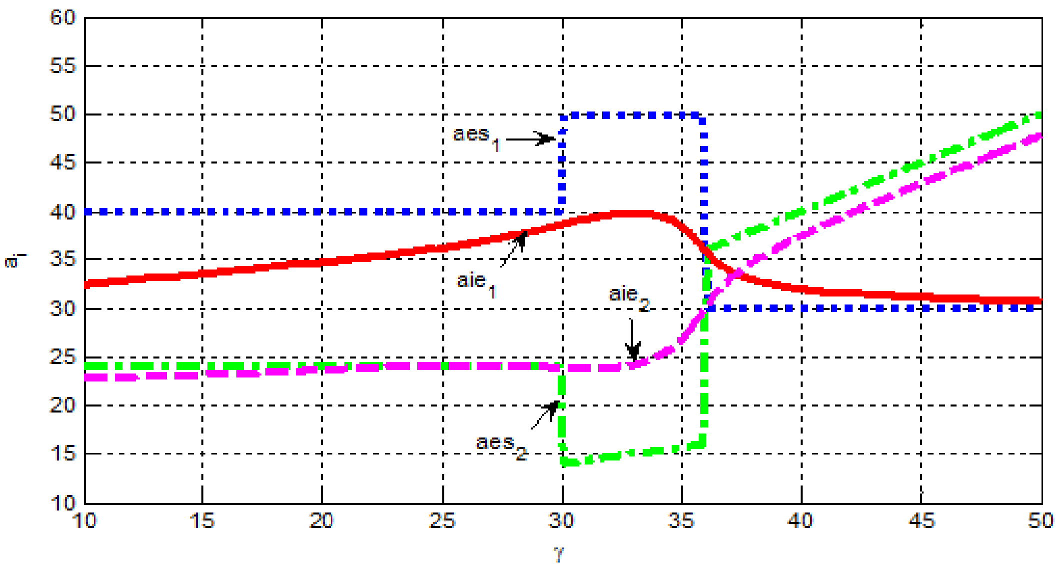

Figure 2. Aumann-Shapley values dynamics with γ based on different risk measures

In addition, Van Gerwald et al. (2012) pointed out that, the amount of risk for X_2 is bigger than X_1, but when $\gamma < 36$, the risk capital allocated to X_2 smaller than that to X_1. This implies that the more risky subportfolio gets allocated less risk capital. It may be unfair. We can see from Figure 2 that Aumann-Shapley values based on Iso-entropic risk measure can not solve this problem, neither.

Another drawback of allocating risk capital based on Aumann-Shapley value is that it can lead to the situation that certain portfolios are allocated a negative risk capital. Though, negative risk capital shows that the corresponding portfolio provides some hedge potential. However, when it is used in a ratio of return over allocated risk capital-type performance measure, it will yield a bad result, because a slightly negative risk capital would yield a largely negative risk adjusted performance measure (Denault, 2001). Let us see another example, which is the same as that in Van Gerwald et al. (2012).

Example 3.2.

Consider a risk capital allocation problem defined by $\Omega = \{\omega_1,...\omega_3\}$, and $\mathbb{N}=\{1,2,3\}$

$$\pi=\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}, X_1=\begin{bmatrix}-5\\25\\-5\end{bmatrix}, X_2=\begin{bmatrix}10\\10\\-5\end{bmatrix}, X_3=\begin{bmatrix}0\\10\\60\end{bmatrix}$$

So

$$X_P=X_1+X_2+X_3=\begin{bmatrix}5\\45\\50\end{bmatrix}$$

We can compute the Aumann-Shapley values for all the three subportfolios based on *ES* and iso-entropic risk measure, at any level, $\alpha = 0.1$ for example:

$$(aes(X_1), aes(X_2), aes(X_3)) = (-5, -5, 60), \text{ and } (aie(X_1), aie(X_2), aie(X_3)) = (-4.6, -4.8, 59.3)$$

We can see that there are negative risk capital allocation both based on expected shortfall and iso-entropic risk measures. Because there are several drawbacks, marginal risk contribution (Aumann-Shapley values) may be not a good allocation rule. So, Van Gerwald et al. (2012) proposed a new allocation rule called Excess based allocation. In next subsection, we will introduce it, and discuss its property, especially this method based on iso-entropic risk measure.

3.3. Excess Based Allocation Rule and Iso-Entropic Coherent Risk Measure

To avoid the draws of marginal risk contribution rule, Van Gerwald et al. (2012) proposed one allocation rule based on excess of loss over the risk capital, called EBA. They point out that because part of the probability distribution of the individual losses is ignored, it can lead to error allocations, which in turn implies that the expected losses of portfolios in excess of the amount of risk capital allocated to them can differ substantially across portfolios. So, they try to find an alternative allocation rule to minimize the excess risks in some rational sense, such as lexicographical ordering.

Suppose that $S \subseteq 2^n$, for an allocation problem $R(X, \pi, \rho)$, a_s is the risk capital allocated to S, and X_S is the loss variable, then the excess of S respect to a_s is defined:

$$e(S, a, R) = E[(X_S - a_s) 1_{X_s > a_s}] \tag{3.12}$$

And then EBA (Excess based allocation) is defined as:

$$EBA = \left\{ a | \theta[\bar{e}(a, R)] \leq_{lex} \theta[\bar{e}(y, R)] \right\} \quad a \in F(R) \, and \, \forall y \in F(R) \tag{3.13}$$

Where $F(R)$ is the set of all feasible risk capital allocations for a risk capital allocation problem R, it requires that $F(R)$ must satisfy Equation (3.12) and $\min_{\omega \in \Omega} X_i(\omega) \leq a_i \leq \rho(X_i)$. And $\bar{e}(a, R)$ is the vector that contains the excesses of all subsets of portfolios $S \subseteq 2^n$ with respect to the allocation a. And $\theta[x]$ is a descending permutation of vector $x = \{x_1, x_2, ...\}$. Equation (3.13) is to determine the feasible allocation that minimizes the excesses in lexicographical sense.

Lexicographical ordering is defined as: for $k \in \mathbb{N}$, and any two vectors $x, y \in \mathbb{R}^k$, x is lexicographically strictly smaller than y, denoted as $x <_{lex} y$ if there exists an $i \leq k$ such that $x_i < y_i$ and for all $j<i$ it holds that $x_i = y_i$. Moreover, x is lexicographically smaller than y, denoted as $x \leq_{lex} y$ if $x = y$ or $x <_{lex} y$.

Van Gerwald et al. (2012) have proved that EBA of Equation (3.13) is unique. And they show that EBA satisfies all the five properties above mentioned. Further, they show that it satisfies continuity on $\mathcal{R}$, which means that when the distribution of loss is discrete, the allocation of EBA is continuous, not as the allocation of CTE. And EBA is not negative, which will be very suitable for performance evaluation.

To solve the allocation problem of EBA easily, Van Gerwald et al. (2012) propose a linear programming approach. Their approach is inspired by an approach to determine the nucleolus of a cooperative game. Before the linear programming, some notations must be defined.

They denote b_{01}, b_{02}, b_1, ..., b_P as follows:

$$\begin{cases} b_{01}=\{i|a_i=\min X_i(\omega)\}, \\ b_{02}=\{i|a_i=\rho(X_i)\}, \\ b_1=\{S|e_S=\max\{e_T|T\subseteq n\}\}, \\ b_j=\{S|e_S=\max\{e_T|T\subseteq n, T\notin b_1\cup\ldots\cup b_{j-1}\}\} \end{cases} \tag{3.14}$$

for every $j\in\mathbb{N}, j\geq 2$ such that $b_1\cup\ldots\cup\ldots b_{j-1}\neq 2^n$. The number $p\in\mathbb{N}$, such that $b_1\cup\ldots\cup\ldots b_P=2^n$, and for all $1 \leq k \leq p$, it holds $b_k \neq \varnothing$. And $e_S = e(S, a, R)$.

Then solving the following linear program problem $p_k(b_1, ..., b_{k-1}, e_1, ..., e_{k-1})$:

$$\begin{aligned} &\min v_k \\ s.t.\ & a_i^k\leq\rho(X_i), i\in n, \\ & a_i^k\geq\min X_i(\omega), i\in n \\ & a_P^k=\rho(X_P), \\ & \sum\pi(\omega)\cdot\lambda_k(S,\omega)=e_j, S\in b_j, j<k, \\ & \sum\pi(\omega)\cdot\lambda_k(S,\omega)\leq v_k, S\in 2^n\backslash(b_1\cup\ldots\cup b_{k-1}), \\ & \lambda_k(S,\omega)\geq X_S(\omega)-a_S^k, S\in 2^n, \\ & \lambda(S,\omega)\geq 0, S\in 2^n, \omega\in\Omega \end{aligned} \tag{3.15}$$

Denote the a feasible solution to the linear program (3.15) by the tuple $a^k, v_k, \lambda_k\in R^N \mathrm{x} R \mathrm{x} R^{m\mathrm{x}2^N}$, and the optimal solution of $p_k(b_1, ..., b_{k-1}, e_1, ..., e_{k-1})$ is $(\bar{a}^k, \bar{v}_k. \bar{\lambda}_k)$. For the optimal solution, denote

$$\begin{aligned} \bar{b}_k&=\{S\in 2^n|e_s=\bar{v}_k\}, \\ \bar{b}_{01}&=\{i\in n|\bar{a}_i^k=\min X_i(\omega)\}, \\ \bar{b}_{02}&=\{i\in n|\bar{a}_i^k=\rho(X_i)\}. \end{aligned} \tag{3.16}$$

Then according to the Lemma 4.3 in Van Gerwald et al. (2012), it shows that if $\bar{v}_k = 0$, then $EBA = \bar{a}^k$; if $\bar{v}_k > 0$, then $EBA_S = \bar{a}^k$ where $S = b_k$.

At last, the algorithm of the linear program problem is given as follows:

(1) Set $k = 1$.

(2) Obtain an optimal solution ($\bar{a}^k$, $\bar{v}_k$, $\bar{\lambda}_k$) to $p_k(\bar{b}_1, ..., \bar{b}_{k-1}, \bar{v}_1, ..., \bar{v}_{k-1})$.

(3) If $\bar{v}_k > 0$ go to Step 4, if $\bar{v}_k = 0$ else stop.

(4) If $\bar{b}_1\cup\ldots\cup\ldots\bar{b}_k\neq 2^n$, let $k = k+1$, else stop.

After Several times of iteration, we can get the final result: $\bar{a} = EBA$.

For the EBA allocation rule, Van Gerwald et al. (2012) proved that if the risk measure used in EBA is a coherent one, it satisfies all the five properties: No Diversification, Riskless Portfolio, Symmetry, Translation Invariance and Scale Invariance. In addition, they pointed out that EBA also has continuous property and it can solve the problem brought by marginal risk contribution rule mentioned above, such as abrupt changes, unfairness and negative values in allocation.

Now, let's use the Examples 3.1 and 3.2 again to show the advantages of EBA. Firstly, let's see Example 3.1, we compute the EBA based on expected shortfall and iso-entropic risk measure. The results are showed in Figure 3. In the figure, EBAES means that EBA based on expected shortfall for X_1 and X_2 are the same $EBAES(X_1) = EBAES(X_2)$ when $\gamma \leq 32.4$, and when $\gamma > 32.4$, $EBAES(X_1) < EBAES(X_2)$; namely $EBAES(X_1) \leq EBAES(X_2)$ for all $\gamma \in R$.

For EBA based on iso-entropic risk measure, the things are the same: when $\gamma \leq 32.4$ $EBAIE(X_1) = EBAIE(X_2)$, denoted by EBAIE; when $\gamma > 32.4$, $EBAIE(X_1) < EBAIE(X_2)$, so $EBAIE(X_1) \leq EBAIE(X_2)$ for all $\gamma \in R$. Because, when $\gamma > 32.4$, $EBAIE(X_2)$ almost equals $EBAES(X_2)$, we use EBA_2 to denote them.

From the Figure 3, we can conclude that: Firstly, both EBA based on expected shortfall and iso-entropic risk measure are continuous, there are no abrupt changes.

Secondly, both methods are consistent with the amount of risk, namely, $\rho(X_1) \leq \rho(X_2) \Rightarrow EBA(X_1) \leq EBA(X_2)$, which means fairness for all the subportfolios.

Thirdly, EBA based on iso-entropic risk measure(EBAIE) is better than EBA based on expected shortfall(EBAES), because it EBAIE is more sensitive and smoother in risk change than EBAES, which is the main contribution of iso-entropic risk measure.

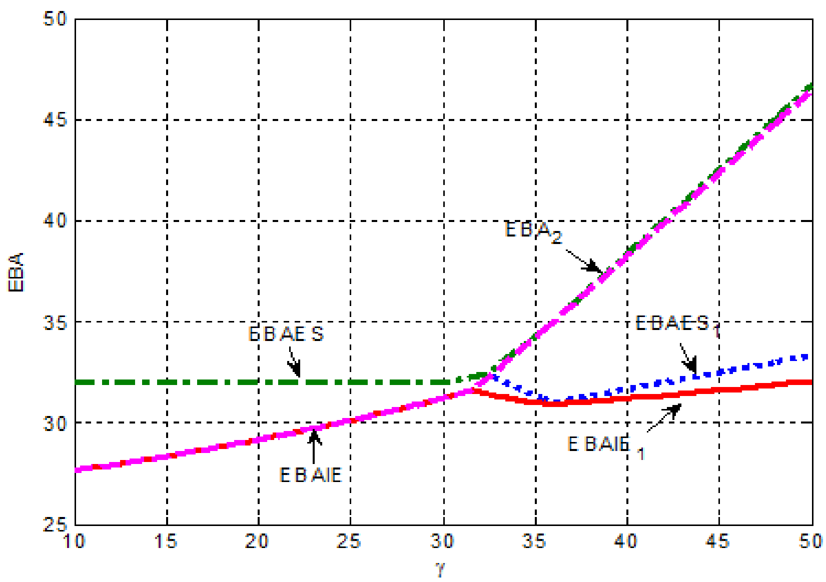

Figure 3. EBA dynamics with based on different risk measures

Now let's see Example 3.2. We compute the EBA based on expected shortfall and iso-entropic risk measure. The results are:

$$(EBAES(X_1), EBAES(X_2), EBAES(X_3)) = (10.06,2.43,37.5)$$

and

$$(EBAIE(X_1), EBAIE(X_2), EBAIE(X_3)) = (10.0,2.47,37.47).$$

We can see that there are no negative value any more. So, Van Gerwald et al. (2012) pointed that it is one of the advantages of EBA compared with marginal risk contribution rule. But for this point, I don't think so. It is because that for any risk capital allocation rule, if it satisfies the property of translation invariance, it can be negative. Translation invariance means that $EBA(X_i + c) = EBA(X_i) + c$, we can always find many c to make $EBA(Y_i) = EBA(X_i + c) < 0$. For this point, negative allocation may be not a drawback.

4. Allocation of Risk Capital Based on Iso-Entropic Coherent Risk Measure in Chinese Market

In this section, we use the risk capital allocation rule based on iso-entropic risk measure to Chinese market. Firstly, we compute the amount of risk for the risk assets, especially the whole portfolio, and check if there are potentials for risk capital allocation. Then, we compute the risk capital allocation based on several allocation rules, and compare the results.

4.1. Computation of risk amount

In this subsection, we build a portfolio of assets and compute their risk amount. We select composite index of Shanghai market (SCI), bonds index of Shanghai market (SBI) and funds index of Shanghai market (SFI) to build the portfolio. The data period is 1.4.2003—30.3.2012, the sample size is 2,187. We take the loss as:

$$x_t = 1 - p_t / p_{t-1}, t = 1, ..., T$$

In which p_t is the price of the real index. Denote the portfolio X_P as

$$X_P = X_{SCI} + (-X_{SBI}) + (-X_{SFI}) \tag{4.1}$$

Where X_{SCI}, X_{SBI} and X_{SFI} are the losses of SCI, SBI and SFI. To show more risk diversified and show negative allocation value, we adjust the proportion of some subportfolios in the portfolio, X_{SBI} and X_{SFI} are negative.

Then we compute the amount of risk for the portfolio and its subportfolios based on several risk measures. As a comparison, the standard error and covariance descriptive are computed too, because we will use them to compute some other risk capital allocations. The computed statistics of all the losses see Table 1. Where i = {*SCI*, *SBI*, *SFI*, *P*}. The confidence level is at α = 0.03, other confidence levels are similar.

statistics	SCI	SBI	SFI	X_P
mean	-3.33E-4	-1.24E-3	-7.46E-4	-1.66E-3
$\sigma(X_i)$	0.0157	0.0021	0.0164	0.0322
$Cov(X_i, X_P)$	-1.76E-6	2.91E-6	11.94E-6	13.09E-6
ES	45.31	7.80	48.44	15.03
Iso-entropic measure	30.79	6.06	34.97	15.22

Table 1. The risk amount of portfolios based on different measures

From Table 1, it is showed that SCI is negative correlation with SBI and SFI, so they can hedge each other, which leads to a large part of the risk are hedged. This can decrease the risk capital of the whole portfolio largely. Apparently, $\rho(X_P) \leq \sum_i^N \rho(X_i)$, it is the source and drive of risk capital allocation. For expected shortfall, $\sum_i^N ES(X_i) - ES(X_P) = 96.52$; and for iso-entropic risk measure, $\sum_i^N IE(X_i) - IE(X_P) = 56.6$.

4.2. Risk Capital Allocation Based on Several Allocation Rules

In this subsection, we introduce several risk capital allocation rules, and then compute the allocation solution and do some analysis. The allocation rules include marginal risk contribution rules based on expected shortfall (*ES*) and iso-entropic risk measures (*IE*), the excess based allocation using expected shortfall (*ES*) and iso-entropic risk measures (*IE*), which are introduced in previous section 3. As comparisons, we use the simple rules, such as proportional allocation rule based on standard error and covariance allocation principle.

Here, the proportional allocation principle is based on standard error:

$$a(X_i) = \frac{\sigma(X_i)}{\sum_{i=1}^N \sigma(X_i)} \rho(X_P) \tag{4.2}$$

Apparently, proportional allocation principle does not satisfy the properties of riskless portfolio and translation invariance.

The covariance allocation rule (Overbeck, 2000 and Dhaene et al., 2012) is as follows:

$$a(X_i) = \frac{\operatorname{cov}(X_i, X_p)}{\operatorname{cov}(X_p, X_p)} \rho(X_p) \quad (4.3)$$

Because *VaR* does not satisfy subadditivity, which is the important property for risk capital allocation, it is not a suitable risk measure for capital allocation. We omit it here. However, it is still be powerful in some fields (Wan, Yang & Wan, 2014). We just compute the capital allocation based on coherent risk measures.

Limited to the length, here we only display the result of allocation at confidence level α = 0.03, the result of other situation is the similar. The risk capital allocations are calculated under six allocation principles above mentioned, see Table 2. The ratio of risk capital for each subportfolios are listed in brackets. From the Table 2, it shows that the result of proportional rule based on standard error and EBAs based on two kinds of coherent risk measures are near, but proportional allocation principle does not satisfy the properties of riskless portfolio and translation invariance. For same reason, covariance allocation rule does not satisfy the properties of riskless portfolio and translation invariance. The EBA based on two kinds of coherent risk measures are consistent with the amount of risk of each subportfolios, which means fairness for all the subportfolios. But Aumann-shapley values and Covariance based allocation are not consistent with the amount of risk of each subportfolios, and there are negative values for them, which is not good for performance. Compare the two coherent risk measures, it shows that the iso-entropic risk measure can display more information than expected shortfall. So, we can conclude that the EBA based on iso-entropic risk measure may be the best one in these allocation rules.

Allocation rule	SCI	SBI	SFI
Proportional rule(standard error)	6.90 (45.9%)	0.92 (6.01%)	7.21 (48.09%)
Covariance based allocation	-2.02 (-13.4%)	3.34 (22.2%)	13.71 (91.2%)
Aumann-shapley-ES based allocation	-5.10 (-33.9%)	3.27 (21.8%)	16.86 (125.5%)
Aumann-shapley-IE based allocation	-4.57 (-30%)	-0.81 (-5.3%)	20.60 (135.3%)
EBA--ES based allocation	6.27 (41.7%)	1.09 (7.3%)	7.67 (51.0%)
EBA--IE based allocation	6.36 (41.8%)	1.10 (7.2%)	7.76 (51.0%)

Table 2. Risk capital allocation under different rules

5. Performance of the Risk Capital Allocation Rules

In section 4, we just show the result of allocation, and compare them with simple property. Here we apply a method to judge the performance of these allocation principles. It is RAROC (Risk adjusted return on capital, James, 1996) method:

$$RAROC_i = \frac{\overline{R}_i}{a_i}, \quad i=1,\ldots,N \tag{5.1}$$

Where $\overline{R}_i$ is the average return of subporfolio *i*. Liu, Yang and Zhou (2014) use similar indicator to evaluate the performance of mutual fund portfolios, it turns this is a good indicator. Here we apply the negative value of mean in Table 1. And then we compare the average $\overline{RAROC} = \frac{1}{N}\sum_{i=1}^{N} RAROC_i$. The results see Table 3. In brackets, it is the percentage reduction in risk capital as compared to the risk capital, namely $1 - \frac{a_i}{\rho(X_i)}$. It is the indicator to judge allocation principles used in Van Gerwald et al (2012). It shows that proportional rule may be the best, if we apply reduction indicator and RAROC indicator; but it doesn't satisfy some properties required. As we know that, proportional rule (standard error) is not suitable for an allocation rule because of its flaws. As it is pointed, when the risk capital allocation is negative, the performance valuation can be meaningless. So we replace the value with a "*", when there is a negative risk capital. The EBA based on two kinds of coherent risk measures may be the best. From the reduction indicator, EBA based on ES is a little higher than EBA based on IE. However, from the RAROC indicator, EBA based on IE is a little higher than EBA based on ES. With one word, combining the advantages of iso-entropic risk measure and EBA, the EBA based on IE is strongly recommended.

Allocation rule	SCI	SBI	SFI	average
Proportional rule(standard error)	0.0483 (0.85)	1.348 (0.88)	0.1035 (0.85)	0.50 (0.86)
Covariance based allocation	* (1.04)	0.371 (0.57)	0.0544 (0.72)	* (0.78)
Aumann-shapley-ES based allocation	* (1.11)	0.379 (0.58)	0.0442 (0.65)	* (0.78)
Aumann-shapley-IE based allocation	* (1.15)	* (1.13)	0.0366 (0.41)	* (0.9)
EBA--ES based allocation	0.0531 (0.86)	1.138 (0.86)	0.0972 (0.84)	0.4294 (0.85)
EBA--IE based allocation	0.0530 (0.79)	1.141 (0.82)	0.0972 (0.78)	0.4304 (0.80)

Table 3. RAROC of capital allocation under six rules

6. Conclusion

There are two sides for a risk capital allocation problem. One is to quantify the risk of the portfolio and its subportfolios with a good risk measure properly. Another is to allocate the risk capital to each subportfolios properly. The former is the base of the later step. So a good risk measure is very important for risk capital allocation. Now, the most fashionable risk is VaR and expected shortfall (ES), but both have some drawbacks. VaR is not coherent, ES is. But ES is only use a small part of information of the distribution, it is not smooth and is only consistent with lower than second-order stochastic dominance, it can not distinguish the risk well. Zheng and Chen (2012) proposed a new coherent risk measure called iso-entropic risk measure. This new risk measure use all the information of the whole distribution, it is smooth, and consistent with higher-order stochastic dominance. It has higher resolution of risk. Based on this new risk measure, we propose two kinds of risk capital allocation methods: one is marginal risk contribution based on iso-entropic risk measure, another is EBA based on iso-entropic risk measure. We discuss their properties comparing with other allocation rules. It shows that the risk capital allocation rules based on iso-entropic risk measure are better than the other rules. And, the EBA based on iso-entropic risk measure is recommended strongly. However, there are some problems are still open. One is how to treat the negative value of allocation. Second is that the consistence between the allocated risk capital and the amount of the risk needs to be studied further.

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References

Acerbi, C., Nordio, C., & Sirtori, C. (2001). *Expected shortfall as a tool for financial risk management*. Working paper available online in: http://arxiv.org/abs/cond-mat/0102304

Acerbi, C., & Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking and Finance*, 26(7), 1487-1503. http://dx.doi.org/10.1016/S0378-4266(02)00283-2

Artzner, P., Delbaen, F., Eber J.M., & Heath D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3), 203-228. http://dx.doi.org/10.1111/1467-9965.00068

Cherny, A.S., & Madan, D.B. (2006). *Coherent measurement of factor risks*. Preprint, Moscow State University. http://dx.doi.org/10.2139/ssrn.904543

Cummins, J.D. (2000). Allocation of capital in the insurance industry. *Risk Management and Insurance Review*, 3(1), 7-27. http://dx.doi.org/10.1111/j.1540-6296.2000.tb00013.x

Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk*, 4(1), 1-34. Available online in: http://neumann.hec.ca/pages/michel.denault/J%20of%20Risk%202001.pdf

Dhaene, J., Henrard, L., Landsman, Z. Vandendorpe, A., & Vanduffel, S. (2008). Some results on the CTE-based capital allocation rule. *Insurance: Mathematics and Economics*, 42(2), 855-863. http://dx.doi.org/10.1016/j.insmatheco.2007.09.004

Dhaene, J., Tsanakas, A., Emiliano, V., & Vanduffel, S. (2012). Optimal Capital Allocation Principles. *The Journal of Risk and Insurance*, 79(1), 1-28. http://dx.doi.org/10.1111/j.1539-6975.2011.01408.x

Fischer, T. (2003). Risk capital allocation by coherent risk measures based on one-sided moments. *Insurance: Mathematics and Economics*, 32(1), 135-146.

http://dx.doi.org/10.1016/S0167-6687(02)00209-3

Fishburn, P.C. (1974). Lexicographic orders, utilities and decision rules: a survey. *Management Science*, 20(11), 1442-1471. http://dx.doi.org/10.1287/mnsc.20.11.1442

James, C.M. (1996). RAROC based capital budgeting and performance evaluation: A case study of bank capital allocation. *Financial Institutions Center, The Wharton School, University of Pennsylvania*, Working Paper 96-40. http://dx.doi.org/10.2139/ssrn.1000

Kalkbrener, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance*, 15(3), 425-437. http://dx.doi.org/10.1111/j.1467-9965.2005.00227.x

Kim, J., & Hardy, M. (2009). A capital allocation based on a solvency exchange option. *Insurance: Mathematics and Economics*, 44(3), 357-366.

http://dx.doi.org/10.1016/j.insmatheco.2008.11.006

Liu, Q., Yang, K., & Zhou, Y. (2014). The effect of management team characteristics on performance and style extremity of mutual fund portfolios. *Journal of Industrial Engineering and Management*, 7(1), 294-310. http://dx.doi.org/10.3926/jiem.987

Ma, C., & Wong, W.K. (2010). Stochastic dominance and risk measure: A decision-theoretic foundation for VaR and C-VaR. *European Journal of Operational Research*, 207(2), 927-935. http://dx.doi.org/10.1016/j.ejor.2010.05.043

Myers, S.C., & Read, J.A. Jr. (2001). Capital allocation for insurance companies. *Journal of Risk and Insurance*, 68(4), 545-580. http://dx.doi.org/10.2307/2691539

Overbeck, L. (2000). Allocation of economic capital in loan portfolios. In J. Franke, G.Stahl, & W. Hardle (Eds), *Measuring Risk in Complex Stochastic Systems, Lecture Notes in Statistics* (147, pp 1-17). New York: Springer.

Panjer, H.H. (2002). Measurement of risk, solvency requirements and allocation of capital within financial conglomerates. Research Report 01–15. *Institute of Insurance and Pension Research*. Available online in:
http://www.actuaries.org/EVENTS/Congresses/Cancun/afir_subject/afir_14_panjer.pdf

Tasche, D. (1999). *Risk contributions and performance measurement*. Working paper, available online in:
http://www.researchgate.net/publication/2378752_Risk_Contributions_and_Performance_Measurement

Tasche, D. (2004). *Allocating portfolio economic capital to sub-portfolios*. Working paper, available online in:
http://www.researchgate.net/publication/228930070_Allocating_portfolio_economic_capital_to_sub-portfolios

Van Gerwald, G., De Waegenaere, A., & Norde, H. (2012). Excess based allocation of risk capital. *Insurance: Mathematics and Economics*, 50(1), 26-42.

http://dx.doi.org/10.1016/j.insmatheco.2011.09.003

Venter, G.G. (2004). Capital allocation survey with commentary. *North American Actuarial Journal*, 8(2), 96-107. http://dx.doi.org/10.1080/10920277.2004.10596139

Wan, X., Yang, S., & Wan, L. (2014). Risk metrics model in purchasing risk measurement. *Journal of System and Management Sciences*, 4(1), 62-71. Available online in: http://www.aasmr.org/jsms/Vol4 /No.1/JSMS_Vol4_No1_008.pdf

Wong, W.K., & Ma, C. (2008). Preferences over location-scale family. *Economic Theory,* 37(1), 119-146. http://dx.doi.org/10.1007/s00199-007-0254-3

Zheng, C., & Chen, Y. (2012). Coherent risk measure based on relative entropy. *Applied Mathematics & Information Sciences*, 6(2), 233-238. Available online in: http:// www.naturalspublishing.com /files/published/k4tjo2z823r7g5.pdf

Zheng, C., & Chen, Y. (2014a). The comparisons for three kinds of quantile-based risk measures. *Information Technology Journal*, 13(6), 1147-1153.

http://dx.doi.org/10.3923/itj.2014.1147.1153

Zheng, C., & Chen, Y. (2014b). Portfolio selection based on relative entropy coherent risk measure. *Systems Engineering - Theory & Practice*, 34(3), 648-655. Available online in: http:// www.sysengi. com/CN/Y2014/V34/I3/648

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